Homework 3

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Spring 2016

1 Expander mixing lemma

Let G be a d-regular graph. Show the following version of the expander mixing lemma: For every set $S \subseteq V(G)$,

$$\mathop{\mathbb{P}}_{uv \in E(G)}\Big\{u \in S \wedge v \in S\Big\} \leq \mathop{\mathbb{P}}_{u \in V(G)}\Big\{u \in S\Big\} \cdot \Big(\mathop{\mathbb{P}}_{v \in V(G)}\Big\{v \in S\Big\} + \lambda(G)\Big). \tag{1}$$

2 Lower bound on support by first two moments

Let X be real-valued random variable such that $\mathbb{E}\, X \geq 0$. Show the following lower bound on the probability of its support in terms of its first two moments,

$$\mathbb{P}\Big\{X > 0\Big\} \geq (\mathbb{E}\, X)^2 / \mathbb{E}\, X^2\,. \tag{2}$$

3 Eigenvectors and eigenvalues of hypercube

Let G be d-regular graph with $V(G) = \mathbb{F}_2^d$ and

$$E(G) = \left\{(x, x + \mathbf{1}_i) \;\middle|\; x \in \mathbb{F}_2^d, i \in [d]\right\}, \tag{3}$$

where $\mathbf{1}_i \in \mathbb{F}_2^d$ denotes the vector with i-th coordinate equal to 1 and all others equal to 0.

1. Show that Fourier basis is an eigenbasis of the adjacency matrix of the graph.
2. Determine $\lambda(G)$.
3. Construct the G' by adding d self-loops to every vertex of G (so that G' is $2d$-regular). Show that $\lambda(G) \leq 1 - \Omega(1/d)$.

Footnotes