Combinatorial vs spectral expansion (Cheeger's Inequality)

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Eigenvalue gaps allow us to upper bound the rate of convergence of random walks on graphs.

In this note, we will look at the connection between the eigenvalue gap and a combinatorial graph parameter, called "expansion""

Let G be a d-regular graph with n vertices. We identify G with its random-walk matrix (normalized adjacency matrix).

Definition: The *eigenvalue gap* of G is defined as

$$\gamma(G) = \min_{f \perp u} \frac{\langle f, L_G f \rangle}{\|f\|_2^2}, \tag{1}$$

where $L_G = I - G$ is the *Laplacian* of G.

Exercise: Check that $\gamma(G) = \lambda_1(G) - \lambda_2(G)$ is indeed the gap between the largest and second largest eigenvalue of G.

We would like to characterize the eigenvalue gap of a graph in terms of a combinatorial property. The right combinatorial property turns out to be expansion, which measures how well G is connected.

Definition: The *expansion* of G is defined as

$$\phi(G) = \min_{S \subseteq V,\ |S| \le n/2} \frac{\frac{1}{d}|E(S, \bar{S})|}{|S|}. \tag{2}$$

(Here, $E(S, \bar{S})$ is the set of edges from S to $\bar{S}$.) The quantity $\phi(S) = \frac{1}{d}|E(S, \bar{S})|/|S|$ is the expansion of S.

Exercise: Check that

$$\langle f, L_G f \rangle = \sum_i \mathop{\mathbb{E}}_{j \sim i} \tfrac{1}{2}(f_i - f_j)^2. \tag{3}$$

(So the Laplacian exactly measures how close the values of f are for a typical edge of the graph.)

It follows that $\langle 1_S, L_G 1_S \rangle = \frac{1}{d}|E(S, \bar{S})|$ and therefore

$$\phi(G) = \min_{S \subseteq V,\ |S| \le n/2} \langle 1_S, L_G 1_S \rangle / \|1_S\|_2^2. \tag{4}$$

Cheeger's inequality asserts that $\gamma(G)$ is the same as $\phi(G)$ up to quadratic factor.

Theorem: *(Cheeger's inequality)*

$$\gamma(G)/2 \le \phi(G) \le 2\sqrt{\gamma(G)}. \tag{5}$$

The following direction is the easier one.

Lemma: $\phi(G) \ge \gamma(G)/2$

Proof: Let S be a subset of vertices with $|S| \le n/2$. We like to lower bound its expansion. Let $p = 1_S/|S|$ be the uniform distribution over S. Then $p = u + f$, where u is the uniform distribution and f is orthogonal to u. Then,

$$\phi(S) = \frac{\langle p, L_G p\rangle}{\|p\|_2^2} = \frac{\langle f, L_G f\rangle}{\|u\|_2^2 + \|f\|_2^2} \ge \frac{\gamma\|f\|_2^2}{\|u\|_2^2 + \|f\|_2^2}. \tag{6}$$

To finish the proof, it is enough to show that $\|f\|_2^2 \ge \|u\|_2^2$. (This means that the distribution p is "far" from u.) Recall that $f = 1_S/|S| - u$. It follows that the distance of $1_S/|S|$ and u decreases as the size of S grows. Hence, the distance is smallest if $|S| = n/2$. In this case, f takes values $\pm 1/n$. Hence, $\|f\|_2^2 = n \cdot 1/n^2 = 1/n = \|u\|_2^2$.

Next, we will prove the other direction of Cheeger's inequality.

The first step is the following lemma (which we have seen in class).

Lemma: Suppose h is a function with $|\mathrm{supp}(h)| \le n/2$ and $\langle h, L_G h\rangle \le \epsilon\|h\|_2^2$. Then there exists a set S with $|S| \le n/2$ and $\phi(S) \le \sqrt{2\epsilon}$.

W.l.o.g. we may assume $0 \le h \le 1$. The main idea is to construct a distribution over sets S by choosing a random threshold $\tau \in [0,1]$ and setting $S = \{i \mid h_i^2 > \tau\}$. It is easy to verify that $\mathbb{E}_S |S| = \|h\|_2^2$. Using Cauchy-Schwarz, one can also show $\mathbb{E}_S \frac{1}{d}|E(S, \bar{S})| \le \sqrt{2\epsilon}\|h\|_2^2$. Hence, there exists a set in the support of the distribution with expansion at most $\sqrt{2\epsilon}$.

The next step is to construct such a function h.

Lemma: There exists a function h with $|\mathrm{supp}(h)| \le n/2$ and $\langle h, L_G h\rangle \le 2\gamma(G)\|h\|_2^2$.

Proof: Let f be a function such that $f \perp u$ and $\langle f, L_G f\rangle \le \gamma(G)\|f\|_2^2$. Let m be the median of this function. (So that f has value larger than m for exactly half of the vertices.) Let $h_1 = (f-m)_+$ be the positive part of $f - m$ and let $h_2 = (f-m)_-$ be the negative part of $f - m$. Notice that both h_1 and h_2 have exactly half of the vertices in their support. We claim that one of the functions h_1, h_2 satisfies also the other condition of the lemma.

Indeed, we can see that $\langle h_1, L_G h_1\rangle \le \langle f, L_G f\rangle$ and $\langle h_2, L_G h_2\rangle \le \langle f, L_G f\rangle$. On the other hand, $\|f\|_2^2 = \|h_1\|_2^2 + \|h_2\|^2$. Hence, we can choose h to be the function h_1 or h_2 with larger norm.

Footnotes