Explicit expander graphs: Zig-Zag product

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We will describe the *zig-zag product* of graphs, which allows us to reduce the degree of a graph while approximately maintaining its eigenvalue gap.

Let G be a regular graph with n vertices and degree D. (Think of n as a growing parameter and D as large but absolut constant.)

Let H be a regular graph with D vertices and degree d. (Think of d as a relatively small absolut constant, e.g., $d = 100$.)

Suppose that H is a very good expander, i.e., eigenvalue gap close to 1. (Since H has only constant size and we know that graphs with these properties exist, we could compute H efficiently by brute force.)

We will consider graphs with vertex set $[n] \times [D]$. We think of this set as n disjoint *clouds* of size D.

The first graph we consider is $I_n \otimes H$, which consists of n disjoint copies of H, one copy per cloud. (The notation $I_n \otimes H$ stems from the fact that the random walk matrix of the graph is the tensor product of the matrix I_n and the random walk matrix of H.)

Next, we consider a graph $\hat{G}$ obtained from G by splitting every vertex into D new vertices, one for each edge. In other words, $\hat{G}$ is a perfect matching on $[n] \times [D]$ such that *contracting* each cloud yields the graph G.[1]

The idea of the zig-zag product is to combine the two graphs $\hat{G}$ and $I_n \otimes H$ to obtain a graph on $[n] \times [D]$ with much smaller degree than G but approximately the same eigenvalue gap.

Why should the graphs $\hat{G}$ and $I_n \otimes H$ be helpful? One good thing is that both graphs have small degrees. In fact, $\hat{G}$ has only degree 1. Another good thing is that from far away (i.e., if we contract the clouds), the graph $\hat{G}$ looks exactly like G (recall that we wanted the new graph to have the same eigenvalue gap as G). In the zig-zag construction, the graph $I_n \otimes H$ allows us to *effectively* contract the clouds, while maintaining small degree.

Definition: The *zig-zag product* of G with H is the graph

$$G \boxtimes H = (I_n \otimes H) \cdot \hat{G} \cdot (I_n \otimes H). \tag{1}$$

The following lemma shows that $G \boxtimes H$ and G have the same eigenvalue gap up to a $\gamma(H)^2$ factor. (If we choose H carefully, then $\gamma(H) \approx 1$.)

Lemma: $\gamma(G \boxtimes H) \geq \gamma(G) \cdot \gamma(H)^2$.

Proof:

We will use the following useful characterization of the eigenvalue gap: A graph has eigenvalue gap at least γ if and only if its random walk matrix is a convex combination of the walk matrix of the complete graph and a matrix with largest eigenvalue at most 1 such that the walk matrix of the complete graph has weight at least γ.

Hence, we can write $H = \gamma_H J_D + (1 - \gamma_H) E_H$ for $\gamma_H = \gamma(H)$ and a matrix E_H with largest eigenvalue at most 1.

Using this decomposition for H, we can see that $G \boxtimes H$ is a convex combination of four matrices,

$$G \boxtimes H = \gamma_H^2 (I_n \otimes J_D) \hat{G} (I_n \otimes J_D) \tag{2}$$

$$+ \gamma_H(1-\gamma_H)(I_n \otimes J_D)\hat{G}(I_n \otimes E_H) \quad (3)$$

$$+ (1-\gamma_H)\gamma_H(I_n \otimes E_H)\hat{G}(I_n \otimes J_D) \quad (4)$$

$$+ (1-\gamma_H)^2(I_n \otimes E_H)\hat{G}(I_n \otimes E_H). \quad (5)$$

All four matrices have eigenvalues at most 1. Hence,

$$G \boxtimes H = \gamma_H^2(I_n \otimes J_D)\hat{G}(I_n \otimes J_D) + (1-\gamma_H^2)E \quad (6)$$

for a matrix E with eigenvalues at most 1.

How does the graph $(I_n \otimes J_D)\hat{G}(I_n \otimes J_D)$ look like? We claim that this graph is essentially G. (The reason is that the multiplications with $(I_n \otimes J_D$ effectively contract the clouds and we already noted that this contraction makes $\hat{G}$ into G.) Formally, $(I_n \otimes J_D)\hat{G}(I_n \otimes J_D) = G \otimes J_D$.[2]

The graph $G \otimes J_D$ has eigenvalue gap $\gamma_G = \gamma(G)$. Hence, we can write it as a convex combination $G \otimes J_D = \gamma_G \cdot J_{Dn} + (1-\gamma_G)E'$ for a matrix E' with eigenvalues at most 1.

It follows that $G \boxtimes H$ is a convex combination of the three matrices J_{Dn}, E, and E' (all with eigenvalues at most 1). The matrix J_{Dn} has weight $\gamma_H^2 \cdot \gamma_G$ in this convex combination. Thus, $G \boxtimes H$ has eigenvalue gap at least $\gamma_H^2 \cdot \gamma_G$.

Footnotes

1. A more concrete way to construct $\hat{G}$ from G is to map every edge e between u and v in G to an edge between to an edge $\hat{e}$ between (u, i) and (v, i) in $\hat{G}$, where i is the index of e for u and j is the index of e for v. (For this construction, we assign an index $i \in [D]$ to every edge incident to a vertex $u \in [n]$ in G.)

2. Here is one way to see this identity without "index battle": How does a random step in the graph $(I_n \otimes J_D)\hat{G}(I_n \otimes J_D)$ look like? Let (v, j) be a random neighbor of a vertex (u, i) in this graph. To go from (u, i) to (v, j) we take a random step first in $(I_n \otimes J_D$, second in $\hat{G}$ and third in $(I_n \otimes J_D)$. The third step guarantees that even conditioned on u, v, and i, the distribution of j is uniform. What is the distribution of v conditioned on u, i, and j? We claim that v is just a random neighbor of u in G. The reason is that in the first step we go to a random vertex in the cloud of u. Every vertex in this cloud uniquely corresponds to one of the outgoing edges of u. Hence, we selected a random edge out of u when taking the second step in $\hat{G}$ (which brings us to the cloud of a random neighbor v of u). It follows that (v, j) conditioned on (u, i) has the same distribution as in the graph $G \otimes J_D$.